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STABILITY AND STABILIZATION OF AUTONOMOUS SYSTEM ORBITS UNDER STOCHASTIC PERTURBATIONS†

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Both the necessary and sufficient conditions for exponential orbital stability in the mean square for periodic motions of stochastic systems are obtained, using the method of orbital Lyapunov functions. From the sufficiency criteria an orbit stabilization method is given.

1. CONSIDER the system of differential equations

$$dx = f(x)\,dt \tag{1.1}$$

where x is an n-dimensional vector and $f(x)$ is a vector function of appropriate dimensions. Suppose $x = \xi(t)$ is a T-periodic solution of system (1.1) that differs from the point of rest, and γ is the phase trajectory of this solution (orbit). Necessary and sufficient conditions for exponential orbital stability, connected with the Andronov–Witt theorem and its analogues [1–4], belong to the first Lyapunov method. In [5] a method was developed that reduced the investigation of the stability of the orbit to an investigation of the stability of the point of rest. The main method of analysing the stability of systems with random perturbations (see [6, 7]) is the second Lyapunov method. For deterministic systems (1.1) a method of orbital Lyapunov functions has been proposed [8] and is extended here to stochastic systems of the form

$$dx = f(x)\,dt + \sum_{r=1}^{m} \sigma_r(x)\,dw_r(t) \tag{1.2}$$

In (1.2) $\sigma_r(x)$ $(r = 1, \ldots, m)$ is a vector function of appropriate dimensions and $w_r(t)$, $(r = 1, \ldots, m)$ is an independent standard Wiener process. It is assumed that the random noise in (1.2) is such that $x = \xi(t)$ remains a T-periodic solution, i.e.

$$\sigma_r(\xi(t)) = 0, \quad 0 \leqslant t < T.$$

Suppose U is a neighbourhood of the orbit γ such that for any point $x \in U$ one can uniquely find a quantity $\vartheta(x)$, $0 \leqslant \vartheta(x) \leqslant T$ for which $\xi(\vartheta(x))$ is the point on the trajectory γ that is nearest to x. It is clear that the vector

$$\Delta(x) = x - \xi(\vartheta(x))$$

is a displacement from the orbit normal to the vector $f[\xi(\vartheta(x))\,f[\xi(\vartheta(x))]$. We assume that there is a neighbourhood U such that this property holds and which is invariant under both system (1.1) and system (1.2). For system (1.1) such a neighbourhood exists if the orbit γ is exponentially orbitally stable. If U is invariant for system (1.1) and the diffusion coefficients $\sigma_r(x)$ $(r = 1, \ldots, m)$ vanish outside some compact set completely contained in U, then U is also invariant under the stochastic system (1.2).

Definition. A periodic solution $\xi(t)$ of system (1.2) is called exponentially orbitally stable in the mean square (EOMS-stable) in an invariant neighbourhood U if there exist $\alpha > 0$, $K > 0$ such that

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$$E\,|\,\Delta(x(t))\,|^2 \leqslant Ke^{-\alpha t}\,|\,\Delta(x_0)\,|^2 \tag{1.3}$$

for any $x_0 \in U$. In (1.3) $x(t)$ is a solution of (1.2) satisfying the initial condition $x(0) = x_0$.

This paper gives both the necessary and sufficient conditions for EOMS-stability, based on the method of orbital Lyapunov functions. The use of a sufficiency criterion enables one to solve the problem of stabilizing periodic motions $\xi(t)$ of system (1.2).

2. An important role in the investigation of the stability of stochastic systems (see [7]) is played by the generating differential operator

$$Lv(x) = \sum_{i=1}^{n} \frac{\partial v(x)}{\partial x_i} f_i(x) + \frac{1}{2} \sum_{r=1}^{m} \sum_{i,j=1}^{n} \frac{\partial^2 v(x)}{\partial x_i \partial x_j} \sigma_{ri}(x)\sigma_{rj}(x) = \tag{2.1}$$

$$= \left(\frac{\partial v(x)}{\partial x}, f(x)\right) + \frac{1}{2} \sum_{r=1}^{m} \left(\sigma_r(x), \frac{\partial^2 v(x)}{\partial x^2} \sigma_r(x)\right)$$

$$\left(\frac{\partial v}{\partial x} = \left[\frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n}\right]^T, \quad \frac{\partial^2 v}{\partial x^2} = \left[\frac{\partial^2 v}{\partial x_i \partial x_j}\right]_{i,j=1}^{n}\right)$$

We introduce the notation

$$V(\tau) = \frac{1}{2}\left[\frac{\partial^2 v(\xi(\tau))}{\partial x_i \partial x_j}\right]_{i,j=1}^{n}$$

$$F(\tau) = \left[\frac{\partial f_i(\xi(\tau))}{\partial x_j}\right]_{i,j=1}^{n}, \qquad S_r(\tau) = \left[\frac{\partial \sigma_{ri}(\xi(\tau))}{\partial x_j}\right]_{i,j=1}^{n}$$

Here $V(\tau)$, $F(\tau)$ and $S(\tau)$ are T-periodic $(n \times n)$ matrices.

The following lemma is an extension to the stochastic case of the corresponding lemma in [8].

Lemma. Suppose that in a neighbourhood U of the orbit γ there is a sufficiently smooth function $v(x)$ satisfying the conditions $v(x) \geqslant 0$, $v(\xi(\tau)) = 0$, $0 \leqslant \tau < T$. Then

$$V(\tau) f(\xi(\tau)) = 0 \tag{2.2}$$

and for $x \in U$

$$v(x) = (x - \xi(\tau))^T V(\tau)(x - \xi(\tau)) + \delta_1(x, \xi(\tau)) \tag{2.3}$$

$$R(x) \triangleq Lv(x) = (x - \xi(\tau))^T W(\tau)(x - \xi(\tau)) + \delta_2(x, \xi(\tau)) \tag{2.4}$$

$$\left(W(\tau) = V'(\tau) + F^T(\tau)V(\tau) + V(\tau)F(\tau) + \sum_{r=1}^{m} S_r^T(\tau)V(\tau)S_r(\tau)\right)$$

and the functions $\delta_i(x, y)$ are such that $|\delta_i(x, y)| \leqslant \beta_i |x - y|^3$, $\beta_i > 0$ $(i = 1, 2)$.

We denote by $P(f)$ the matrix corresponding to the projection operator onto the subspace orthogonal to the vector $f \neq 0$; $P(f) = I - |f|^{-2} ff^T$ where I is the unit matrix. We put $P_\tau = P(f(\xi(\tau)))$. We shall call the quadratic form $x^T Ax$, and also the symmetric matrix A, $P(f)$-positive definite, and write

$$A \overset{P(f)}{>} 0$$

($P(f)$-non-negative definite and write

$$A \overset{P(f)}{\geqslant} 0)$$

if for any vector $x \neq 0$ orthogonal to the vector f, the inequality $x^T Ax > 0$ $(x^T Ax \geqslant 0)$ is satisfied.

Theorem 1. Suppose that for some T-periodic P_τ-positive definite matrix $C(\tau)$ there exists a T-periodic positive definite matrix $V(\tau)$ such that

$$\dot{V}(\tau) + F^T(\tau)V(\tau) + V(\tau)F(\tau) + \sum_{r=1}^{m} S_r^T(\tau)V(\tau)S_r(\tau) = -P_\tau C(\tau)P_\tau \tag{2.5}$$

Suppose the diffusion coefficients $\sigma_r(x)$ of system (1.2) vanish outside the r-tube $U_r = \{x: \Delta^T(x)V(\vartheta(x))\Delta(x) < r\}$ for sufficiently small r. Then a T-periodic solution $\xi(t)$ of system (1.2) is EOMS-stable in the set U_{r_1} for some $r_1 > r$.

If a T-periodic solution $\xi(t)$ of system (1.2) is EOMS-stable in some invariant neighbourhood U and the integral $E\int_0^\infty |\Delta(x(s))|^2 ds$ is a sufficiently smooth function in U, then for any T-periodic P_τ-positive definite matrix $C(\tau)$ there exists a T-periodic P_τ-positive definite matrix $V(\tau)$ satisfying Eq. (2.5).

Proof of sufficiency. Suppose $V(\tau)$ is a matrix satisfying the conditions of the theorem. There exists some $r_0 > 0$ for which the function $\vartheta(x)$ is defined in the domain U_{r_0}. Then the function $v(x) + \Delta^T(x)V(\vartheta(x))\Delta(x)$ is also defined in U_{r_0}. From the lemma [putting $\tau = \vartheta(x)$ in (2.4)] and (2.5) there follows the relation

$$Lv(x) = -\Delta^T(x)C(\vartheta(x))\Delta(x) + \delta_2(x, \xi(\vartheta(x))) \tag{2.6}$$

In view of the $P_{\vartheta(x)}$-positive definiteness of the matrices $V(\vartheta(x))$ and $C(\vartheta(x))$ one can find positive numbers m, M and α such that

$$\begin{aligned} m\,|\Delta(x)|^2 &\leqslant v(x) \leqslant M\,|\Delta(x)|^2 \\ \alpha\,|\Delta(x)|^2 &\leqslant \Delta^T(x)C(\vartheta(x))\Delta(x) \end{aligned} \tag{2.7}$$

From (2.7) it follows that

$$-\Delta^T(x)C(\vartheta(x))\Delta(x) \leqslant -\alpha M^{-1}v(x) \tag{2.8}$$

The following inequalities are obtained from the lemma and (2.7)

$$|\delta_2(x, \xi(\vartheta(x)))| \leqslant \beta_2\,|\Delta(x)|^3 \leqslant \beta_2 m^{-1}\,|\Delta(x)|\,v(x) \tag{2.9}$$

From (2.6), (2.8) and (2.9) we obtain the inequality

$$Lv(x) \leqslant (\beta_2 m^{-1}|\Delta(x)| - \alpha M^{-1})v(x)$$

which is valid for U_{r_0}.

One can always find an $r_1 \leqslant r_0$ such that in $U_{r_1} \subseteq U_{r_0}$ we have the inequality

$$\beta_2 m^{-1}|\Delta(x)| - \alpha M^{-1} \leqslant -\tfrac{1}{2}\alpha M^{-1}$$

from which it follows in turn that

$$Lv(x) \leqslant -\tfrac{1}{2}\alpha M^{-1}v(x) \tag{2.10}$$

We will now assume that the diffusion coefficients of system (1.2) vanish outside U_r for some $r < r_1$. In this case the domain U_{r_1}, being invariant for the deterministic system (1.1) (which follows from the fact that V is a Lyapunov function for the deterministic system), also remains invariant for the stochastic system (1.2). From Ito's formula we obtain

$$\frac{d}{dt}[Ev(x(t))] = ELv(x(t)) \tag{2.11}$$

From (2.10) and (2.11) it follows that for any $x_0 \in U_{r_1}$ the inequality

$$Ev(x(t)) \leqslant \exp(-\tfrac{1}{2}\alpha M^{-1}t)Ev(x_0)$$

is satisfied.

Finally, using (2.7), we obtain

$$E\,|\Delta(x(t))|^2 \leqslant Mm^{-1}\exp(-\tfrac{1}{2}\alpha M^{-1}t)E\,|\Delta(x_0)|^2$$

The EOMS-stability is proved.

Necessity. In view of the EOMS-stability of system (1.2) in U the function

$$v(x) = E\int_0^\infty \Delta^T(x(s))C(\vartheta(x(s)))\Delta(x(s))ds$$

is defined where $x(t)$ is the solution of system (1.2) with initial condition $x(0) = x$. From the P_τ-positive definiteness of the matrix $C(\tau)$ it follows that the function $v(x)$ satisfies the conditions of the lemma, while the matrix $V(\tau) = ½[\partial^2 v[\xi(\tau)]/\partial x_i \partial x_j]_{i,j=1}^n$ is P_τ-positive definite. Because

$$Ev(x(t)) - v(x) = -E\int_0^t \Delta^T(x(s))C(\vartheta(x(s)))\Delta(x(s))\,ds$$

we have

$$d/dt\, Ev(x(t)) = -\Delta^T(x)C(\vartheta(x))\Delta(x) \tag{2.12}$$

On the other hand, from Ito's formula and (2.4) it follows that

$$d/dt\, Ev(x(0)) = ELv(x) = \Delta^T(x)W(\vartheta(x))\Delta(x) + \delta_2(x, \xi(\vartheta(x))) \tag{2.13}$$

where

$$W(\tau) = V^{\cdot}(\tau) + F^T(\tau)V(\tau) + V(\tau)F(\tau) + \sum_{r=1}^m S_r^T(\tau)V(\tau)S_r(\tau)$$

It follows from (2.12) and (2.13) that $P_\tau W(\tau)P_\tau = -P_\tau C(\tau)P_\tau$. Because $P_\tau W(\tau)P_\tau = W(\tau)$, equality (2.5) holds. The necessity is proved.

Theorem 2. Suppose that the matrix $C(\tau)$, instead of satisfying the assumption of P_τ-positive definiteness as in Theorem 1, satisfies the conditions

$$C(\tau) - \alpha(\tau)I \overset{P_\tau}{\geqslant} 0 \tag{2.14}$$

$$\int_0^T \alpha(\tau)d\tau > 0 \tag{2.15}$$

where $\alpha(\tau)$ is some T-periodic function. Then EOMS-stability holds.

Proof. In Theorem 1 the case $\alpha(\tau) \geqslant \alpha > 0$ was considered. We now reduce the more general case (2.15) to the one previously considered. For this it is sufficient to construct from the matrix $V(\tau)$ of Theorem 2 a matrix $Z(\tau)$ satisfying the equality

$$Z^{\cdot}(\tau) + F^T(\tau)Z(\tau) + Z(\tau)F(\tau) + \sum_{r=1}^m S_r^T(\tau)Z(\tau)S_r(\tau) = -P_\tau D(\tau)P_\tau \tag{2.16}$$

with a matrix $D(\tau)$ such that for some $\mu > 0$ the matrix $D(\tau) - \mu I$ is P_τ-non-negative definite. We shall construct the matrix $Z(\tau)$ in the form $Z(\tau) = \rho(\tau)V(\tau)$, where $\rho(\tau) > 0$ is a differentiable T-periodic function. In view of Eq. (2.5) and the equality $V(\tau) = P_\tau V(\tau)P_\tau$ relation (2.16) will be satisfied if one puts

$$D(\tau) = -\rho'(\tau)V(\tau) + \rho(\tau)C(\tau)$$

From (2.14) there follows the inequality

$$D(\tau) \overset{P_\tau}{\geqslant} -\rho'(\tau)V(\tau) + \rho(\tau)\alpha(\tau)I$$

Furthermore, it follows from (2.7) that

$$D(\tau) \overset{P_\tau}{\geqslant} [-\rho'(\tau) + \alpha(\tau)M^{-1}\rho(\tau)]\, V(\tau) \overset{P_\tau}{\geqslant} m[-\rho'(\tau) + \alpha(\tau)M^{-1}\rho(\tau)]I \tag{2.17}$$

Suppose $\rho(\tau)$ satisfies the differential equation

$$\rho'(\tau) - \alpha(\tau)M^{-1}\rho = -k\rho \tag{2.18}$$

where k is some constant. The solution of this equation is the positive function

$$\rho(t) = \exp \int_0^t (\alpha(s)M^{-1} - k)ds$$

From the T-periodicity requirement we obtain for k the equation

$$1 = \exp \int_0^T (\alpha(s)M^{-1} - k)\,ds$$

from which we find

$$k = \frac{1}{TM} \int_0^T \alpha(s)ds$$

From (2.17) and (2.18) we obtain the inequality

$$D(\tau) \overset{P_\tau}{\geqslant} mk\rho(\tau)I \overset{P_\tau}{\geqslant} mk \min_{[0,T]} \rho(\tau)I$$

i.e. for $\mu > 0$ one can take $\mu = mk\min_{[0,T]}\rho(\tau)$. Theorem 2 is proved.

Previously [9] the problem of the stability of the rest point of a complex stochastic system with several sources of noise was reduced to finding the value of some criterion computed for a simpler system with a smaller number of noise sources (and in particular, for a deterministic system). The possibilities of such an approach for investigations of EOMS-stability are demonstrated in the following theorem.

Theorem 3. Suppose that a T-periodic solution $\xi(\tau)$ of the deterministic system (1.1) is exponentially orbitally stable, and the diffusion coefficients of system (1.2) vanish outside a sufficiently small neighbourhood of U_r, while inside U_r they satisfy the inequality

$$\sum_{r=1}^{m} |\sigma_r(x)|^2 \leqslant \mu(\vartheta(x))\Delta^T(x)\Gamma(\vartheta(x))\Delta(x) \tag{2.19}$$

where $\mu(\tau) \geqslant 0$ is a T-periodic function, and $\Gamma(\tau)$ is a T-periodic P_τ-positive definite matrix. Suppose $V(\tau)$, a T-periodic P_τ-positive definite matrix, is a solution of the deterministic Lyapunov equation

$$V^{\cdot}(\tau) + F^T(\tau)V(\tau) + V(\tau)F(\tau) = -P_\tau\Gamma(\tau)P_\tau \tag{2.20}$$

Then the inequality

$$\frac{1}{T}\int_0^T \mu(s)\,\mathrm{tr}\,V(s)ds < 1 \tag{2.21}$$

is a sufficient condition for EOMS-stability of the solution $\xi(\tau)$ of system (1.2) in U_{r_1} for some $r_1 > r$.

Proof. Suppose $v(x) = \Delta^T(x)V(\vartheta(x))\Delta(x)$, where $V(\tau)$ is a T-periodic P_τ-positive definite matrix of the solution of the deterministic Lyapunov equation (2.20). (We note that in view of the EO-stability, such a matrix always exists.) With condition (2.19) for the function $R(x) = Lv(x)$ there exists the majorant

$$\bar{R}(x) = \left(\frac{\partial v(x)}{\partial x}, f(x)\right) + \mu(\vartheta(x))\Delta^T(x)\Gamma(\vartheta(x))\Delta(x)\,\mathrm{tr}\left[\frac{1}{2}\frac{\partial^2 v(x)}{\partial x^2}\right]$$

Furthermore, using, as in the lemma, the expansions for $[\partial v(x)/\partial x, f(x)]$ and $\frac{1}{2}\partial^2 v(x)/\partial x^2$, we obtain

$$\bar{R}(x) = \Delta^T(x)\bar{W}(\vartheta(x))\Delta(x) + \delta(x, \xi(\vartheta(x)))$$

where

$$\bar{W}(\tau) = V^{\cdot}(\tau) + F^T(\tau)V(\tau) + V(\tau)F(\tau) + \mu(\tau)P_\tau\Gamma(\tau)P_\tau \text{ tr } V(\tau)$$
$$|\delta(x, \xi(\vartheta(x)))| \leqslant \beta\,|\Delta(x)|^3, \qquad \beta > 0$$

From the inequality $R(x) \leqslant \bar{R}(x)$ there follows the inequality

$$W(\tau) \overset{P_T}{\leqslant} \bar{W}(\tau),$$

from which $C(\tau) = -W(\tau)$ and $\bar{C}(\tau) = -\bar{W}(\tau)$ we obtain

$$C(\tau) \overset{P_T}{\geqslant} \bar{C}(\tau).$$

Because $\bar{C}(\tau) = [1 - \mu(\tau) \operatorname{tr} V(\tau)]P_\tau\Gamma(\tau)P_\tau$, we have

$$C(\tau) \overset{P_T}{\geqslant} (1 - \mu(\tau) \text{ tr } V(\tau))P_\tau\Gamma(\tau)P_\tau.$$

By the P_τ-positive definiteness of the matrix $\Gamma(\tau)$ one can find a positive number ν such that

$$P_\tau\Gamma(\tau)P_\tau \overset{P_T}{\geqslant} \nu I.$$

Thus, $C(\tau)$ satisfies the inequality (2.14) in which $\alpha(\tau) = \nu[1 - \mu(\tau) \operatorname{tr} V(\tau)]$. Here condition (2.15) follows directly from (2.21). Hence by Theorem 2 the solution $\xi(\tau)$ of system (1.2) is EOMS-stable in a neighbourhood U_{r_1} for some $r_1 > r$. The theorem is proved.

3. Consider the problem of stabilizing the periodic motion of a system with control

$$dx = (f(x) + B(\vartheta(x))u)\,dt + \sum_{r=1}^{m} \sigma_r(x)\,dw_r(t) \tag{3.1}$$

where $B(\tau)$ is a T-periodic $(n \times k)$ matrix and u is the k-dimensional control.

Theorem 4. Suppose that the control is (3.1) is constructed in the form

$$u = -K(\vartheta(x))\Delta(x) \tag{3.2}$$

with a weighting matrix

$$K(\tau) = R^{-1}(\tau)B^T(\tau)V(\tau)$$

where $R(\tau)$ is a T-periodic positive definite matrix of dimensions $k \times k$ and $V(\tau)$ is a solution of the matrix Riccati equation

$$V^{\cdot}(\tau) + F^T(\tau)V(\tau) + V(\tau)F(\tau) - V(\tau)B(\tau)R^{-1}(\tau)B^T(\tau)V(\tau) + \tag{3.3}$$
$$+ \sum_{r=1}^{m} S_r^T(\tau)V(\tau)S_r(\tau) = -P_\tau C(\tau)P_\tau$$

Suppose $V(\tau)$ and $C(\tau)$ satisfy the conditions of Theorem 1. Then, if the diffusion coefficients in (3.1) vanish outside U_r for sufficiently small r, the orbit γ of system (3.1) with control (3.2) is EOMS-stable.

The proof follows directly from Theorem 1.

It has been shown that in the deterministic case similar stabilizing controls were close to optimal in the problem of minimizing some functional. It seems that here too the regulator (3.2), (3.3) is close to optimal for the functional

$$J = E\int_0^\infty [\Delta^T(x)C(\vartheta(x))\Delta(x) + u^T R(\vartheta(x))u]\,dt$$

4. In the following examples we take

$$v(x) = \Delta^T(x)G(\vartheta(x))\Delta(x)$$

where $G(\tau)$, $0 \leqslant \tau < T$ is a T-periodic non-negative definite matrix. One can show that for this function

$$V(\tau) = G(\tau) - \frac{2}{f^T(\tau)f(\tau)} G(\tau)f(\tau)f^T(\tau) + \frac{f^T(\tau)G(\tau)f(\tau)}{[f^T(\tau)f(\tau)]^2} f(\tau)f^T(\tau)$$

In particular, for $G = I$

$$V(\tau) = I - \frac{1}{f^T(\tau)f(\tau)} f(\tau)f^T(\tau) = P_\tau$$

We give the formula for

$$W \triangleq V + F^T V + VF + \sum_{r=1}^{m} S_r^T V S_r \quad \text{for} \quad G = I.$$

We have

$$W = \frac{1}{f^T f}[ff^T(F+F^T) + (F+F^T)ff^T] - (F+F^T) - \frac{f^T(F+F^T)f}{(f^T f)^2} ff^T - \sum_{r=1}^{m} S_r^T [I - \frac{ff^T}{f^T f}] S_r \tag{4.1}$$

One can show that

$$W(\tau)P_\tau = W(\tau) \tag{4.2}$$

In view of (4.2), one obtains for C in relation (2.5) the matrix $C = -W$.

Example 1. We consider the Van der Pol equation with multiple noise sources, written in the form of the system

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = -x_1 + \epsilon x_2(1 - x_1^2) + \sigma(x_1, x_2)w \tag{4.3}$$

As we know, the asymptotically stable orbit $x = \xi(\tau)$ for the deterministic Van der Pol equation for small $\epsilon > 0$ differs little from a circle of radius 2, and this is used in later calculations. Suppose

$$\sigma(x_1, x_2) = \mu(x_1 - \xi_1(\vartheta(x))) \tag{4.4}$$

where $\mu > 0$ is some constant, so that the orbit investigated is a solution of system (4.3). The intensity of the noise is governed by μ. We find a quantity μ_0 such that for $\mu < \mu_0$ the orbit is EOMS-stable (and here, in accordance with the results obtained, it is assumed that the noise is of the form (4.4) sufficiently near to the orbit, and vanishes outside some tube surrounding the orbit).

Taking as $C(\tau)$ the matrix $-W(\tau)$ in accordance with (4.1), we estimate (Cx, x) for a vector x orthogonal to $f(\tau)$ with norm $|x| = 1$. We obtain

$$(Cx, x) = 2\epsilon\xi_2^2(2\xi_1^2 - 1) - \tfrac{1}{4}\mu^2\xi_1^2 + \xi_2^2 + O(\epsilon^2) + O(\epsilon\mu^2)$$

If $\alpha(\tau)$ is put equal to the value of (Cx, x) obtained, the matrix $C(\tau) - \alpha(\tau)I$ will be P_τ-non-negative definite (more precisely, P_τ-null). Using the fact that $\xi_1 = 2\cos\tau + O(\epsilon)$, $\xi_2 = -2\sin\tau + O(\epsilon)$ and $T = 2\pi + O(\epsilon)$, we find

$$\int_0^T \alpha(\tau)d\tau = 4\pi\epsilon - \pi\mu^2 + O(\epsilon^2) + O(\epsilon\mu^2)$$

Hence, if $\mu^2 < 4\epsilon$, then for sufficiently small ϵ the orbit of system (4.3) will be EOMS-stable. Therefore $\mu_0 = 2\sqrt{\epsilon}$.

Example 2. We again consider system (4.3), but now with a function σ of the form

$$\sigma(x_1, x_2) = \mu\,|\Delta(x_1, x_2)| = \mu[(x_1 - \xi_1(\vartheta))^2 + (x_2 - \xi_2(\nu))^2]^{1/2} \tag{4.5}$$

near the orbit.

The derivatives

$$\partial\sigma(\xi(\vartheta))/\partial x_1, \qquad \partial\sigma(\xi(\vartheta))/\partial x_2$$

do not exist for this function.

However, the results of the preceding sections can, without any special difficulties, be carried over to system (1.2), in which the coefficients of the noise have the form $\sigma_r(x) = \varphi_r\alpha_r(\rho_r)$, where φ_r is a vector, $\rho_r = [(x-\xi(\vartheta(x)))^T\Gamma_r(\vartheta(x))(x-\xi(\vartheta(x)))]^{1/2}$ is a scalar, Γ_r is a non-negative definite matrix, and $\alpha_r(\rho)$ is a smooth scalar function for $\rho \geq 0$, not equal to zero except that at zero $\alpha_r(0) = 0$. In this case all the results of Sec. 2 remain as before with Eq. (2.5) replaced by the equation

$$W \triangleq V^{\cdot} + F^T V + VF + \sum_{r=1}^{m} (\alpha_r'(0))^2 (\varphi_r^T V \varphi_r) P_T \Gamma_r P_T = -P_T C P_T \tag{4.6}$$

In investigating the stability and stabilization of the points of rest similar noise sources (noise of the second type) were considered [9].

For system (4.3), (4.5) $m = 1$, $\Gamma = I$, the vector $\varphi = (0, \mu)^T$, $\alpha'(0) = 1$. Taking as $v(x)$ the same function as in example 1, we again estimate (Cx, x), using the equality $C = -W$ (relation (4.2) being satisfied here as well), where W is found from (4.6). We obtain

$$(Cx, x) = 2\epsilon\xi_2^2(3\xi_1^2 - 1) - \mu^2\xi_2^2 + O(\epsilon^2) + O(\epsilon\mu^2)$$

Here for an appropriate $\alpha(\tau)$

$$\int_0^T \alpha(\tau)d\tau = 4\pi\epsilon - 4\pi\mu^2 + O(\epsilon^2) + O(\epsilon\mu^2)$$

and so

$$\mu_0 = \sqrt{\epsilon}.$$

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